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Solutions of the Jin-Schmidt Equation in Lucas Numbers

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Abstract

In this paper, we study the integer solutions of the following equation that is so called the Jin-Schmidt equation,

$$AX^2 + BY^2 + CZ^2 = DXYZ + 1,$$

where $(X, Y, Z) = (L_i, L_j, L_k)$, with $i, j, k \geq 1$ such that L_i, L_j and L_k represent terms in the Lucas sequence that is defined by the relation $L_0 = 2$, $L_1 = 1$, $L_{n+1} = L_n + L_{n-1}$ with $n \geq 1$.

Keywords: Diophantine equations; Lucas sequences; Jin-Schmidt equation; Magma software; elliptic curve equation.

1 Introduction

Suppose that f is an equation with the variables $x_1, x_2, \dots, x_n$ and $n \geq 2$, then,

$$f(x_1, x_2, \dots, x_n) = 0,$$

is called a Diophantine equation if the unknowns are integers. Here, we recall some well-known Diophantine equations, for example, the Fermat's equation,

$$x^n + y^n = z^n,$$

has no positive integer solutions x and y with $n > 2$. In fact, the elliptic curve equation is an important Diophantine equation, that has the form,

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where $a_1, \dots, a_6 \in \mathbb{C}$ and the discriminant Δ, is given by:

$$\begin{aligned}
\Delta &= -\gamma_2^2\gamma_8 - 8\gamma_4^3 - 27\gamma_6^2 + 9\gamma_2\gamma_4\gamma_6,\\
\gamma_2 &= a_1^2 + 4a_2,\\
\gamma_4 &= 2a_4 + a_1a_3,\\
\gamma_6 &= a_3^2 + 4a_6,\\
\gamma_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2.
\end{aligned}$$

The elliptic curve equation has many applications especially in cryptography and other fields in mathematics and sciences as it was applied by Al-Saffar in [1, 2]. Another well known Diophantine equation is known as the Markoff equation, that has the form,

$$X^2 + Y^2 + Z^2 = 3XYZ, \tag{1}$$

where X, Y and Z are positive integers, with $X \leq Y \leq Z$. This equation were studied by the scientist Markoff in the year of 1879–1880 [11, 12] as he found that the set of its solution is as follows:

$$(X, Y, Z) \in \{(1, 1, 1), (X, Z, 3XZ - Y), (Y, Z, 3YZ - X)\}.$$

This set gives an infinite number of solutions, called Markoff triples. Markoff showed that there is one-to-one correspondence between the Markoff triples and the indefinite quadratic forms with minimal greater than $\frac{1}{3}\sqrt{\Delta_1}$, where Δ_1 is the discriminant of the indefinite quadratic forms. This equation was also expanded and studied by the scientist Rosenberger [14] in 1979, which has the form,

$$AX^2 + BY^2 + BZ^2 = DXYZ, \tag{2}$$

where

$$(A, B, C, D) \in \{(1, 2, 3, 6), (1, 1, 1, 3), (1, 1, 1, 1), (1, 1, 2, 2), (1, 1, 5, 5), (1, 1, 2, 4)\},$$

with $A, B, C, D \in \mathbb{N}$ and $gcd(A, B) = gcd(A, C) = gcd(C, D) = 1$ and $A, B, C \setminus D$. Rosenberger named this equation as Markoff-Rosenberger equation and showed that it has infinitely many solutions. After that, the scientists Jin and Schmidt studied the following expansion of the Markoff-Rosenberger equation (called by the Jin-Schmidt equation) in 2001 [8] such that $X, Y, Z \in \mathbb{N}$:

$$AX^2 + BY^2 + CZ^2 = DXYZ + 1, \tag{3}$$

where they noticed that (3) has infinitely many solutions only with,

$(A,B,C,D) \in \{(2,1,2,2),(2,2,3,6),(3,1,6,6),(5,1,5,5),(6,10,15,30),(7,2,14,14),(1,m,m,2m)\}$,

where m is positive integer. In case of $(A,B,C,D) = (2,2,3,6)$, the equation has many applications. For example, it is connected to the constant for quaternions and constants for complex numbers on the circle $\left\{t \in \mathbb{C} \mid |t| = \frac{1}{\sqrt{2}}\right\}$ the field $\mathbb{Q}(\sqrt{-3})$. In fact, recently many authors have made Diophantine equations more interesting by studying their solutions in special linear recurrence sequences, that are defined as follows.

Assume that $\{W_n\}$ is a sequence of the form,

$$W_{n+d} = a_1 W_{n+d-1} + a_2 W_{n+d-2} + \ldots + a_d W_n, \tag{4}$$

for all $n \geq 0$ and $a_1, a_2, \ldots, a_d \in \mathbb{C}$ with $a_d \neq 0$ (where d is the order of the sequence).

The sequence (4) is called a binary linear recurrence sequence if $d = 2$, and it is also called a ternary linear recurrence sequence if $d = 3$. An example of a binary linear recurrence sequences is the Fibonacci sequence that is defined by,

$$F_n = F_{n-1} + F_{n-2},$$

with $n \geq 2$, $F_0 = 0$, $F_1 = 1$. For all $n \geq 0$, the terms of $\{F_n\}$ can be obtained by the following Binet's formula:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{where} \quad (\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right).$$

Note that α is called the golden number and $\beta = \frac{-1}{\alpha}$. Another example of a binary linear recurrence sequence is the Lucas sequence that is defined by,

$$L_n = L_{n-1} + L_{n-2},$$

where $L_0 = 2$, $L_1 = 1$, and $n \geq 2$. The terms of $\{L_n\}$ can be obtained by thy Binet's formula [13]:

$$L_n = \alpha^n + \beta^n, \quad \forall n \geq 0. \tag{5}$$

The terms of Lucas sequence satisfy the identity,

$$\alpha^{n-1} \leq L_n \leq \alpha^{n+1} \text{ holds for all } n \geq 1. \tag{6}$$

The terms of $\{F_n\}$ and $\{L_n\}$ satisfy the equation,

$$L_k^2 = 5F_k^2 \pm 4. \tag{7}$$

In the late of 19th century, the French scientist, Edouard Anatole Lucas [10] around the year 1842–1891 introduced the Lucas sequence, which was named after him. The Lucas sequence is considered as the Fibonacci sequence except with different initials. They both are represented by the same golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$. Through this relationship, it becomes clear that both sequences are linked by common properties. This sequence has a great application in the Lucas-Lehmer test, that is used to discover and verify large prime numbers [3].

A well known example of the ternary linear recurrence sequences is so called by the Tribonacci sequence that's defined by,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

where $T_0 = 0, T_1 = T_2 = 1$ for all $n \geq 3$.

Indeed, the solutions to Diophantine equations in linear recurrence sequences is the main of interest for many researchers. In the following, we recall some such results starting with the result of Hashim [5] in which he solved the following equation completely:

$$\frac{1}{V_n(P_2, Q_2)} = \sum_{k=1}^{\infty} \frac{V_{k-1}(P_1, Q_1)}{x^k},$$

where $\{V_n(P, Q)\}$ denotes the Lucas sequence of the second kind for certain nonzero relatively prime integers P and Q. Also, Hashim and Tengely [6] solved certain Diophantine equations represented by reciprocals of Lucas sequences.

An important example of these interesting studies concerning Markoff equation and its generalization was initiated in 2018 by Luca and Srinivasan [9] for studying the relationship between Diophantine equations and linear recurrence sequences by determining the solutions of the Markoff equation in Fibonacci sequence. Namely, they determined the solutions of (1) where X, Y, and Z belong to the Fibonacci sequence.

Another interesting study was given by Tengely [16] in 2020 in which he investigated the solutions of (2) where the unknowns are Fibonacci numbers. One more important study was given by Hashim and Tengely [7] in 2020 in which they investigated the solutions of (3), where X, Y and Z are terms in Fibonacci numbers and

$$(A, B, C, D) \in T = \{(2, 2, 3, 6), (2, 1, 2, 2), (3, 1, 6, 6), (5, 1, 5, 5), (6, 10, 15, 30), (7, 2, 14, 14)\}.$$

In this article, we study the solutions of (3), (namely the Jin-Schmidt equation) where $(X, Y, Z) = (L_i, L_j, L_k)$ with $i, j, k \geq 1$. That would be studying the solutions of the following equations:

$$2L_i^2 + 2L_j^2 + 3L_k^2 = 6L_iL_jL_k + 1, \tag{8}$$
$$2L_i^2 + L_j^2 + 2L_k^2 = 2L_iL_jL_k + 1, \tag{9}$$
$$3L_i^2 + L_j^2 + 6L_k^2 = 6L_iL_jL_k + 1, \tag{10}$$
$$5L_i^2 + L_j^2 + 5L_k^2 = 5L_iL_jL_k + 1, \tag{11}$$
$$6L_i^2 + 10L_j^2 + 15L_k^2 = 30L_iL_jL_k + 1, \tag{12}$$
$$7L_i^2 + 2L_j^2 + 14L_k^2 = 14L_iL_jL_k + 1, \tag{13}$$

with $i, j, k \geq 1$.

2 Main Approach

In this section, we present the main approach for solving the Jin-Schmidt equation (3) completely, where $(X, Y, Z) = (L_i, L_j, L_k)$ with $i, j, k \geq 1$. Our main approach is based on solving the

Jin-Schmidt equation for each $(A, B, C, D) \in T$ (i.e (8)–(13)) by considering all the possible cases $X \le Y \le Z$, $X \le Z \le Y$, $Y \le X \le Z$, $Y \le Z \le X$, $Z \le X \le Y$, $Z \le Y \le X$. Solving the Jin-Schmidt equation completely at certain (A, B, C, D) requires solving it at all the cases $X \le Y \le Z$, $X \le Z \le Y$, $Y \le X \le Z$, $Y \le Z \le X$, $Z \le X \le Y$, $Z \le Y \le X$, where $(X, Y, Z) = (L_i, L_j, L_k)$ with $i, j, k \ge 1$. In fact, without loss of generality, the condition $(L_i \le L_j \le L_k)$ with $i, j, k \ge 1$ means $i \le j \le k$. So, if we first obtain the solutions of the equation,

$$AL_i^2 + BL_j^2 + CL_k^2 = DL_iL_jL_k + 1, \tag{14}$$

with $(k \ge j \ge i \ge 1)$, then for simplicity, the other cases are solved by considering the latter equation with permuting the components of the first three components in (A, B, C, D). So, the first step is obtaining all the distinct equations of (3) by the permutation of the first three components in each of the tuples $(A, B, C, D) \in T$. We denote each of these distinct equation by,

$$aL_i^2 + bL_j^2 + cL_k^2 = dL_iL_jL_k + 1, \tag{15}$$

with $1 \le i \le j \le k$.

In the following, we give the main steps for determining the solution (L_i, L_j, L_k) with $1 \le i \le j \le k$ of (15).

Step 1: We determine an upper bound for i in (15). We reformulate it as the following:

$$cL_k - dL_iL_j = -\frac{aL_i^2 + bL_j^2}{L_k} + \frac{1}{L_k}. \tag{16}$$

By substituting (5) in L_i, L_j, and L_k in (16), we get that,

$$c\alpha^k - d\alpha^{i+j} = -\frac{aL_i^2 + bL_j^2}{L_k} + \frac{1}{L_k} - c\beta^k + d(\alpha^i\beta^j + \alpha^j\beta^i + \beta^{i+j}). \tag{17}$$

From inequality (6) and $1 \le i \le j \le k$; (or, $1 \le L_i \le L_j \le L_k$), we get the inequality,

$$\left| -\frac{aL_i^2 + bL_j^2}{L_k} \right| \le (a+b)\frac{L_j^2}{L_k} \le (a+b)\alpha^{2j-k} \le (a+b)\alpha^j, \tag{18}$$

$$\left| \frac{1}{L_k} \right| \le 1 < \alpha^j, \tag{19}$$

$$\left| -c\beta^k \right| = \left| c\alpha^{-k} \right| \le c\alpha^{-j} \le c\alpha^j, \tag{20}$$

$$\left| d(\alpha^i\beta^j + \alpha^j\beta^i + \beta^{i+j}) \right| \le d(2\alpha^j + 1) \le 3d\alpha^j. \tag{21}$$

Taking the absolute values to (17) and plugging (18)–(21) into the right hand side of (17), we get that,

$$\left| c\alpha^k - d\alpha^{i+j} \right| < (1 + a + b + c + 3d)\alpha^j. \tag{22}$$

Multiplying (22) with $\dfrac{1}{c\alpha^{i+j}}$ leads to,

$$\left| \alpha^{k-i-j} - \frac{d}{c} \right| < \frac{h}{\alpha^i}, \tag{23}$$

such that $h = \dfrac{1}{c}(1 + a + b + c + 3d)$. We assume that,

$$\min_{n \in \mathbb{Z}} \left| \alpha^n - \frac{d}{c} \right| = B_1 > 0,$$

so, inequality (23) will be,

$$\alpha^i < \frac{h}{B_1}.$$

Hence, we get the upper bound of i as follows,

$$i \leq \left\lfloor \frac{\ln(\frac{h}{B_1})}{\ln(\alpha)} \right\rfloor = l, \tag{24}$$

where l is a positive integer.

Step 2: We next obtain for $k-j$ in (15) an upper bound. Since the first three components of every tuple (a,b,c,d) of (15) are obtained from the permutations of (A,B,C) in the tuples of the set T. As $15 \geq a,b,c \geq 1$ and $30 \geq d = D \geq 1$, then $d/c \in \{1,2,3,5,6,7\}$ and this implies that $h < 117$. Hence, from (23) we have that,

$$\left| \left|\alpha^{k-i-j}\right| - \left|\frac{d}{c}\right| \right| \leq \left|\alpha^{k-i-j} - \frac{d}{c}\right| < \frac{117}{\alpha} < 72.31, \quad \text{as} \quad i \geq 1.$$

Then,

$$\left|\alpha^{k-i-j}\right| < 72.31 + \left|\frac{d}{c}\right| < 72.31 + 7 < 79.31, \quad \text{as} \quad d/c \leq 7,$$

which gives that,

$$k - j < i + \frac{\ln(79.31)}{\ln(\alpha)} < l + 10.$$

As $i \leq l$, we obtain that,

$$k \leq j + l + 9. \tag{25}$$

Step 3: We reduce the number of the values of $i \in [1,l]$ in (15). This is achieved by determining the values of i, with which the following equation has solutions for y and z,

$$aL_i^2 + by^2 + cz^2 - dL_i yz - 1 = 0,$$

by using the SageMath software's function `solve_ Diophantine()` [15].

Step 4: Finally, for each i remained from Step 3, we investigate the corresponding values of j and k (with $j \leq k \leq j + I + 9$) with which (15) is satisfied. That would be done by firstly writing (15) as follows:

$$bL_j^2 - sL_j + w = 0, \tag{26}$$

such that $s = dL_iL_k$ and $w = aL_i^2 + cL_k^2 - 1$. Also, identity (7) can be written as,

$$(5F_k)^2 = 5L_k^2 \pm 20. \tag{27}$$

Because (26) is a quadratic equation, we solve it as follows:

$$L_j = \frac{s \pm \sqrt{s^2 - 4bw}}{2b},$$

which can be written as,

$$(2bL_j - s)^2 = s^2 - 4bw. \tag{28}$$

Multiplications of (27) with (28) gives the equation,

$$Y_1^2 = (5X_1^2 \pm 20)(d^2L_i^2X_1^2 - 4b(aL_i^2 + cX_1^2 - 1)), \tag{29}$$

with $X_1 = L_k$ and $Y_1 = 5F_k(2bL_j - dL_iL_k)$. Equation (29) is an elliptic curve equation that can be solved using the Magma software [4] with the function `SIntegralLjunggrenPoints()`. Indeed, (29) can be written in the form,

$$Y_1^2 = A_1X_1^4 + B_1X_1^2 + C. \tag{30}$$

So, to calculate the points of (30), we use the Magma software with the function `SIntegralLjunggrenPoints`$([1, A_1, B_1, C_1], [])$.

From every obtained solution (L_i, L_j, L_k) of (15), we acquire the corresponding solutions $(X, Y, Z) = (L_i, L_j, L_k)$ of (14) by comparing the positions of the components (a, b, c, d) and (A, B, C, D), respectively.

3 Main Results

Suppose that S is the set of all distinct tuples obtained by the permutation of components A, B and C in each of the tuples of T.

Theorem 3.1. *If* $(a, b, c, d) \in S$ *and,*

$$B_1 = \min_{I\in\mathbb{Z}} \left|\alpha^I - \frac{d}{c}\right| \neq 0,$$

then $B_1 \geq 0.145$. *Moreover, if* $X = L_i, Y = L_j$ *and* $Z = L_k$ *with* $1 \leq i \leq j \leq k$ *is a solution of* (3), *then* $i \leq 13$ *and* $j \leq k \leq j + 22$.

Proof. At the beginning, we want to prove the value of $B_1 \geq 0.145$. From the set S, we get $d/c \in \{1, 2, 3, 5, 6, 7\}$. In case of $I = 0$ and $B_1 \neq 0$, we have that $d/c \in \{2, 3, 4, 5, 6, 7\}$. Therefore, $B_1 \geq 1$. Now, we assume that $I \leq -1$, then $\alpha^I \leq \alpha^{-1} = \dfrac{2}{1+\sqrt{5}}$. So $B_1 \geq 0.381$. Also, if $I \geq 5$, then $\alpha^5 \geq 11.09$. Therefore, $B_1 \geq 4.09$. If $I = 1$, then $B_1 = \min_{I=1}\left|\alpha^1 - \dfrac{d}{c}\right| \geq \left|\alpha^1 - 2\right| \geq 0.381$. Now, if $I = 2$, then $B_1 = \min_{I=2}\left|\alpha^2 - \dfrac{d}{c}\right| \geq \left|\alpha^2 - 3\right| \geq 0.381$. Also, if $I = 3$, then $B_1 = \min_{I=3}\left|\alpha^3 - \dfrac{d}{c}\right| \geq \left|\alpha^3 - 5\right| \geq 0.763$. Also, for $I = 4$, we get $B_1 = \min_{I=4}\left|\alpha^4 - \dfrac{d}{c}\right| \geq \left|\alpha^4 - 7\right| \geq 0.145$. Based on the above and for all $I \in \mathbb{Z}$, then $B_1 = \min_{I\in\mathbb{Z}}\left|\alpha^I - \dfrac{d}{c}\right| \geq \left|\alpha^4 - 7\right| \geq 0.145$. Now, we want to prove the last part of the Theorem. Using the value of $B_1 \geq 0.145$ in inequality (24), we get the following:

$$i \leq \left|\frac{\ln\left(\dfrac{117}{0.145}\right)}{\ln(\alpha)}\right| < 14.$$

Therefore, $i \leq 13$. Finally, from the inequality (25) and the condition $1 \leq i \leq j \leq k$, we get $j \leq k \leq j + 22$. ∎

Theorem 3.2. *If $(X,Y,Z) = (L_i, L_j, L_k)$ is a solution of (3) where $(A,B,C,D) \in T$, the following table shows the complete set of its solutions:*

Table 1: The complete solutions of (3) in $\{L_n\}$.

Eq.	(A,B,C,D)	$\{(X,Y,Z)\}$
(8)	$(2,2,3,6)$	$\{(1,1,1),(1,7,3),(1,7,11),(7,1,3),(7,1,11)\}$
(9)	$(2,1,2,2)$	$\{\}$
(10)	$(3,1,6,6)$	$\{(1,4,1),(1,4,3),(7,4,1)\}$
(11)	$(5,1,5,5)$	$\{\}$
(12)	$(6,10,15,30)$	$\{(1,1,1),(1,7,3),(1,7,11),(4,1,1),(4,1,7),(4,11,1)\}$
(13)	$(7,2,14,14)$	$\{\}$

Proof. We follow the steps used in the Main Approach (Section 2) and Theorem 3.1 to obtain the solutions of (3) given in Theorem 3.2 and prove them.

Case 1: Suppose that $(A,B,C,D) = (2,2,3,6)$. Using permutations of the coefficients of (8), we get the equations:

$$2L_i^2 + 2L_j^2 + 3L_k^2 = 6L_iL_jL_k + 1, \tag{31}$$
$$2L_i^2 + 3L_j^2 + 2L_k^2 = 6L_iL_jL_k + 1, \tag{32}$$
$$3L_i^2 + 2L_j^2 + 2L_k^2 = 6L_iL_jL_k + 1, \tag{33}$$

where $1 \le i \le j \le k$. From Theorem 3.1, we get that $i \le 13$ and $k - j \le 22$ in all equations. Let us consider (31). We want to find (L_i, L_j, L_k) with $1 \le i \le 13$ and $j \le k \le j + 22$ with which (31) is satisfied. We first follow Step 3 for eliminating the values of i to get that $i \in \{1,4\}$ such that,

$$2L_i^2 + 2y^2 + 3z^2 - 6L_iyz - 1 = 0,$$

is solvable. If $i = 1$ and $j \le k \le j + 22$, then,

$$2L_j^2 - 6L_kL_j + 3L_k^2 + 1 = 0.$$

The latter equation is a quadratic equation with respect to j. Now, we follow Step 4, to find the values of j. We first substitute the value of $i = 1$ and $(a,b,c,d) = (2,2,3,6)$ in the elliptic curve (29) to get the values of k. Then, we get the values of j with $k \in \{j, \ldots, .j + 22\}$. For $i = 1$, we have the elliptic curves,

$$Y_1^2 = 60X_1^4 + 200X_1^2 - 160, \tag{34}$$
$$Y_1^2 = 60X_1^4 - 280X_1^2 + 160. \tag{35}$$

With the Magma function `SIntegralLjunggrenPoints()`, we solve the above equations. We are indeed interested in the values of X_1 as $X_1 = L_k$. For (34), we get that $X_1 \in \{\pm1, \pm2, \pm11\}$. But we notice that $\pm2, -1, -11 \notin \{L_n\}$ for all $n \ge 1$. For (35), we get $X_1 \in \{\pm2, \pm3\}$. Also $\pm2, -3 \notin \{L_n\}$ for all $n \ge 1$. Finally, from (34) and (35), we get that $L_k \in \{1,3,11\}$. If $L_k = 1$, then $k = 1$. Now, we want to find the values of j from the value of $j \le k \le j + 22$ as follows:

- If $k = j$, then $j = 1$. Therefore, $(L_i, L_j, L_k) = (L_1, L_1, L_1) = (1,1,1)$ and this triple satisfy (31).

- If $j+1 \leq k \leq j+22$, then $j<1$, which contradicts $1=i \leq j \leq k$.

But if $L_k = 3$, then $k = 2$. Similarly, we get the values of j as follows:

- If $k = j$, then we get that $j = 2$. Therefore, $(L_i, L_j, L_k) = (L_1, L_2, L_2) = (1, 3, 3)$ and this triple does not satisfy (31).
- If $k = j+1$, then $j = 1$. Hence, $(L_i, L_j, L_k) = (L_1, L_1, L_2) = (1, 1, 3)$ and $(1, 1, 3)$ does not satisfy (31).
- Finally, if $j+2 \leq k \leq j+22$, then $j < 1$. This is not possible since $1 = i \leq j \leq k$.

Lastly, if $L_k = 11$, then $k = 5$. Hence,

- If $k = j$, then $j = 5$. Thus, $(L_i, L_j, L_k) = (L_1, L_5, L_5) = (1, 11, 11)$ and (31) is not satisfied at this triple.
- If $k = j+1$, then $j = 4$. Therefore, $(L_i, L_j, L_k) = (L_1, L_4, L_5) = (1, 7, 11)$ and this triple satisfy (31).
- If $k = j+2$, then $j = 3$. Hence, $(L_i, L_j, L_k) = (L_1, L_3, L_5) = (1, 4, 11)$ and $(1, 4, 11)$ does not satisfy (31).
- If $k = j+3$, then $j = 2$. So, $(L_i, L_j, L_k) = (L_1, L_2, L_5) = (1, 3, 11)$ and this triple does not satisfy (31).
- If $k = j+4$, then $j = 1$. Subsequently, $(L_i, L_j, L_k) = (L_1, L_1, L_5) = (1, 1, 11)$ and (31) does not hold at this triple.
- Finally, in case of $j+5 \leq k \leq j+22$, we get $j < 1$. This is not possible since $j \geq i \geq 1$.

The same step can be applied in case of $i = 4$ with $k \in \{j, \dots, j+22\}$ such that $j \geq 4$ (since $4 = i \leq j \leq k$). By substituting the values of $L_i = L_4 = 7$ and $(a, b, c, d) = (2, 2, 3, 6)$ in (29), we get,

$$Y_1^2 = 8700X_1^4 + 30920X_1^2 - 15520, \tag{36}$$
$$Y_1^2 = 8700X_1^4 - 38680X_1^2 + 15520, \tag{37}$$

with $X_1 = L_k$ and $Y_1 = 5F_k(4L_j - 42L_k)$. For (36), we obtain that $X_1 \in \{\pm 11\}$. We get that $X_1 \in \{\pm 2, \pm 3\}$ for (37). But, we notice that $\pm 2, \pm 3, -11 \notin \{L_n\}$ for all $n \geq 4$ since $4 = i \leq j \leq k$. Therefore, we have that only $L_k = 11$, then $k = 5$. Now, we investigate the values of j with $j \leq k \leq j+22$ as the following:

- If $k = j$, then $j = 5$. Hence, $(L_i, L_j, L_k) = (L_4, L_5, L_5) = (7, 11, 11)$ and (31) is not satisfied at this triple.
- If $k = j+1$, then $j = 4$. However, $(L_i, L_j, L_k) = (L_4, L_4, L_5) = (7, 7, 11)$ is not satisfying (31).
- Finally, if $k \in \{j+2, \dots, j+22\}$, then $j \leq 3$. It is not possible since $j \geq i = 4$.

Now, we consider (32) and by Theorem 3.1, we obtain $i \leq 13$ and $k - j \leq 22$. We follow Step 3 to eliminate the values of i. We obtain,

$$2L_i^2 + 3y^2 + 2z^2 - 6L_iyz - 1 = 0,$$

is solvable only with $i \in \{1, 4\}$. Next, we follow Step 4 to get values of j and k. For $i = 1$, we get the elliptic curves,

$$Y_1^2 = 60X_1^4 + 180X_1^2 - 240, \tag{38}$$
$$Y_1^2 = 60X_1^4 - 300X_1^2 + 240. \tag{39}$$

For the curve (38), the possible values of X_1, that represent Lucas numbers are given by $X_1 = 1$. Also, with respect to the curve (39), we get that $X_1 \in \{1, 7\}$. Therefore, $L_k \in \{1, 7\}$. If $L_k = 1$, then $k = 1$. Now, we can find the values of j from $j \leq k \leq j+22$ as follows:

- If $k = j$, then $j = 1$. Therefore, $(L_i, L_j, L_k) = (1, 1, 1)$ and this triple satisfy (32).
- Finally, we get $j < 1$ in case of $j+1 \leq k \leq j+22$, then it is not possible because $1 = i \leq j \leq k$.

Lastly, if $L_k = 7$, then $k = 4$. Hence,

- If $k = j$, then $j = 4$. Subsequently, $(L_i, L_j, L_k) = (L_1, L_4, L_4) = (1, 7, 7)$ and $(1, 7, 7)$ does not satisfy (32).
- If $k = j + 1$, then $j = 3$. But, $(L_i, L_j, L_k) = (L_1, L_3, L_4) = (1, 4, 7)$ this triple does not satisfy (32).
- If $k = j + 2$, then $j = 2$. Therefore, $(L_i, L_j, L_k) = (L_1, L_2, L_4) = (1, 3, 7)$ and (32) is satisfied at this triple.
- If $k = j + 3$, then $j = 1$. Subsequently, $(L_i, L_j, L_k) = (L_1, L_1, L_4) = (1, 1, 7)$ and this triple does not satisfy (32).
- Finally, if $j + 4 \leq k = 4 \leq j + 22$, so $j < 1$. This is not possible because $j \geq 1$.

Now, we get the following curves in case of $i = 4$:

$$Y_1^2 = 8700X_1^4 + 28980X_1^2 - 23280, \tag{40}$$
$$Y_1^2 = 8700X_1^4 - 40620X_1^2 + 23280. \tag{41}$$

For the curve (40), the possible values of X_1, that represent Lucas numbers are given by $X_1 = 1$. For the curve (41), we get that $X_1 = 2$. Therefore, $L_k \in \{1, 2\}$. which gives that $k \leq 1$, this is a contradiction as $4 = i \leq j \leq k$. Then (32) has no solution when $i = 4$.

Now, we study (33) and by Theorem 3.1, we get that $1 \leq i \leq 13$ and $j \leq k \leq j+22$. From Step 3, we achieve that $i \in \{1, 2, 5\}$. Following the elliptic curve approach explained in Step 4 in case of $i = 1$, we obtain the equation,

$$Y_1^2 = 100X_1^4 + 320X_1^2 - 320, \tag{42}$$
$$Y_1^2 = 100X_1^4 - 480X_1^2 + 320, \tag{43}$$

with $X_1 = L_k$ and $Y_1 = 5F_K(4L_j - 12L_k)$. From (42) we get that $X_1 \in \{\pm 1\}$ and from (43) we obtain that $X_1 \in \{\pm 2\}$. But $-1, \pm 2 \notin \{L_n\}$ for all $n \geq 1$. Therefore, we only have $X_1 = L_k = 1$, that gives $k = 1$. In the following, we investigate the values of j such that $j \leq k \leq j + 22$.

- If $k = j$, then $j = 1$. Subsequently, $(L_i, L_j, L_k) = (1, 1, 1)$ is a solution to (33).
- In case of $j + 1 \leq k = 1 \leq j + 22$, then $j < 1$, which is not possible as $j \geq 1$.

Similarly, for $i = 2$, we obtain the elliptic curves,

$$Y_1^2 = 1540X_1^4 + 5120X_1^2 - 4160, \tag{44}$$
$$Y_1^2 = 1540X_1^4 - 7200X_1^2 + 4160. \tag{45}$$

We have that $X_1 \in \{\pm 1\}$ for (44) and $X_1 \in \{\pm 2, \pm 7\}$ for (45). We notice that only $7 \in \{L_n\}$ for all $n \geq 2$. If $L_k = 7$, then $k = 4$. Now, we want to find the values of j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 4$. We get that $(L_i, L_j, L_k) = (L_2, L_4, L_4) = (3, 7, 7)$ and this triple does not satisfy (33).
- If $k = j + 1$, then $j = 3$. Hence, $(L_i, L_j, L_k) = (L_2, L_3, L_4) = (3, 4, 7)$ and $(3, 4, 7)$ does not satisfy (33).

- If $k = j + 2$, then $j = 2$. Therefore, $(L_i, L_j, L_k) = (L_2, L_2, L_4) = (3, 3, 7)$ and (33) does not hold at this triple.
- Finally, if $k \in \{j + 3, \dots, j + 22\}$, then $j \leq 1$. This is not possible because $2 = i \leq j \leq k$.

If $i = 5$, using the same way to find $X_1 = L_k$, we get that $X_1 \in \{\pm 1, \pm 2, \pm 7\}$. We notice that of these values of L_k lead to $k < 5$, and this is a contradiction since $5 = i \leq j \leq k$. Therefore, (33) has no solution.

By gathering all of the obtained solutions of (31)–(33) with permuting their components so that they satisfy (8), we get the solutions of (8) as follows:

$$(L_i, L_j, L_k) \in \{(1, 1, 1), (1, 7, 3), (1, 7, 11), (7, 1, 3), (7, 1, 11)\}.$$

Case 2: If $(A, B, C, D) = (2, 1, 2, 2)$. We obtain the distinct equations for (9) by permuting the coefficients of (9). That leads to the following equations:

$$2L_i^2 + L_j^2 + 2L_k^2 = 2L_iL_jL_k + 1, \tag{46}$$
$$L_i^2 + 2L_j^2 + 2L_k^2 = 2L_iL_jL_k + 1, \tag{47}$$
$$2L_i^2 + 2L_j^2 + L_k^2 = 2L_iL_jL_k + 1, \tag{48}$$

where, $1 \leq i \leq j \leq k$. From Theorem 3.1, we get that $i \leq 13$ and $j \leq k \leq j + 22$ in all of (46)–(48). Now, we consider the solutions of (46). By Step 3, we have that,

$$2L_i^2 + y^2 + z^2 - 2L_iyz - 1 = 0,$$

can be solved with respect to y and z only with $i = 3$. By Step 4, we substitute $L_i = L_3 = 4$ and $(a, b, c, d) = (2, 1, 2, 2)$ in (29), we get that,

$$Y_1^2 = 280X_1^4 + 500X_1^2 - 2480, \tag{49}$$
$$Y_1^2 = 280X_1^4 - 1740X_1^2 + 2480. \tag{50}$$

Equation (49) has no integer solutions and $X_1 \in \{\pm 2\}$ is the X-coordinate of the solutions of (50). But, we notice that $\pm 2 \notin \{L_n\}$ for all $n \geq 1$. Therefore, (46) has no solution.

Also, we study the solutions (47). We follow Step 3 to eliminate the value of i. We have that,

$$L_i^2 + 2y^2 + 2z^2 - 2L_iyz - 1 = 0,$$

is solvable only with $i = 2$. Next, we follow Step 4 to get values of j and k. For $i = 2$, we get the elliptic curves,

$$Y_1^2 = 100X_1^4 + 80X_1^2 - 1280, \tag{51}$$
$$Y_1^2 = 100X_1^4 - 720X_1^2 + 1280. \tag{52}$$

For the curve (51), the possible values of X_1, that represent Lucas numbers are given by $X_1 = 4$ and (52) has no Lucas numbers solutions. Therefore, we get that only $L_k = 4$, then $k = 3$. Now, we want to find the values of j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 3$. So, $(L_i, L_j, L_k) = (L_2, L_3, L_3) = (3, 4, 4)$ and (47) is not satisfied at this triple.

- If $k = j + 1$, then $j = 2$. Therefore, $(L_i, L_j, L_k) = (L_2, L_2, L_3) = (3, 3, 4)$ and (47) is not satisfied at this triple.
- If $j + 3 \leq k = 3 \leq j + 22$, then $j \leq 1$, which is not possible since $j \geq 2$.

Finally, we study the solutions of (48). By Step 3, we obtain that,

$$2L_i^2 + 2y^2 + z^2 - 2L_i yz - 1 = 0,$$

has a solution only at $i = 3$. If $i = 3$, we get that the elliptic curves,

$$Y_1^2 = 280X_1^4 - 120X_1^2 - 4960, \tag{53}$$

$$Y_1^2 = 280X_1^4 - 2360X_1^2 + 4960. \tag{54}$$

Equation (53) has no integer solutions and $X_1 \in \{\pm 2 \pm 3\}$ is the X-coordinate of the solutions of (54). But, we notice that $\pm 2, \pm 3 \notin \{L_n\}$ for all $n \geq 3$. Therefore, (48) has no solution. In the end, we notice that (46)–(48) do not have solutions. Therefore, (9) has no solution.

Case 3: If $(A, B, C, D) = (3, 1, 6, 6)$. We obtain the distinct equations for (10) by permuting the coefficients of (10). That leads to the following equations:

$$3L_i^2 + L_j^2 + 6L_k^2 = 6L_iL_jL_k + 1, \tag{55}$$

$$L_i^2 + 3L_j^2 + 6L_k^2 = 6L_iL_jL_k + 1, \tag{56}$$

$$L_i^2 + 6L_j^2 + 3L_k^2 = 6L_iL_jL_k + 1, \tag{57}$$

$$6L_i^2 + L_j^2 + 3L_k^2 = 6L_iL_jL_k + 1, \tag{58}$$

$$6L_i^2 + 3L_j^2 + L_k^2 = 6L_iL_jL_k + 1, \tag{59}$$

$$3L_i^2 + 6L_j^2 + L_k^2 = 6L_iL_jL_k + 1, \tag{60}$$

where, $1 \leq i \leq j \leq k$. From Theorem 3.1, we get that $i \leq 13$ and $k - j \leq 22$ in all (55)–(60). Now, we study the solutions of (55). By Step 3, we get that,

$$3L_i^2 + y^2 + 6z^2 - 6L_i yz - 1 = 0,$$

is solvable only at $i \in \{1, 4, 7\}$. By Step 4, if $i = 1$, we substitute $i = 1$ and $(a, b, c, d) = (3, 1, 6, 6)$ in (29), we get that,

$$Y_1^2 = 60X_1^4 + 200X_1^2 - 160, \tag{61}$$

$$Y_1^2 = 60X_1^4 - 280X_1^2 + 160. \tag{62}$$

We obtain that $X_1 \in \{\pm 1, \pm 2, \pm 11\}$ as solutions for (61) and $X_1 \in \{\pm 2, \pm 3\}$ as solutions for (62). We notice that only $1, 3, 11 \in \{L_n\}$ for all $n \geq 2$. If $L_k = 1$, then $k = 1$. Now, we want to find the values of j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 1$. So, $(L_i, L_j, L_k) = (L_1, L_1, L_1) = (1, 1, 1)$ and this triple does not satisfy (55).
- In case of $j + 1 \leq k = 1 \leq j + 22$, we have that $j < 1$. This is not possible since $1 = i \leq j \leq k$.

If $L_k = 3$, then $k = 2$. From $j \leq k \leq j + 22$, we get the values of j as follows:

- If $k = j$, then $j = 2$. So, $(L_i, L_j, L_k) = (L_1, L_2, L_2) = (1, 3, 3)$, and $(1, 3, 3)$ does not satisfy (55).
- If $k = j + 1$, then $j = 1$. Therefore, $(L_i, L_j, L_k) = (L_1, L_1, L_2) = (1, 1, 3)$ and (55) does not hold at this triple.

- If $j+2 \leq k = 2 \leq j+22$, then $j < 1$. It is not possible since $1 = i \leq j \leq k$.

If $L_k = 11$, then $k = 5$. So, the values of j are obtained as follows:

- If $k = j$, then $j = 5$. So, $(L_i, L_j, L_k) = (L_1, L_5, L_5) = (1, 11, 11)$ and this triple does not satisfy (55).
- If $k = j+1$, then $j = 4$. Therefore, $(L_i, L_j, L_k) = (L_1, L_4, L_5) = (1, 7, 11)$ and this triple does not satisfy (55).
- If $k = j+2$, then $j = 3$. Thus, $(L_i, L_j, L_k) = (L_1, L_3, L_5) = (1, 4, 11)$ and (1,4,11) does not satisfy (55).
- If $k = j+3$, then $j = 2$. However, $(L_i, L_j, L_k) = (L_1, L_2, L_5) = (1, 3, 11)$ does not satisfy (55).
- If $k = j+4$, then $j = 1$. So, $(L_i, L_j, L_k) = (L_1, L_1, L_5) = (1, 1, 11)$ and (55) does not hold at this triple.
- If $j+5 \leq k = 5 \leq j+22$, then $j < 1$. It is not possible since $j \geq 1$.

For $i = 4, 7$, we find that $X_1 = L_k \in \{\pm 1, \pm 2\}$ when $i = 4$, and $X_1 = L_k \in \{\pm 1, \pm 2\}$ when $i = 7$. However, the corresponding values of L_k lead to $k \leq 1$, and this is a contradiction since $4, 7 = i \leq j \leq k$. Then, (55) has no solutions at $i = 4, 7$.

Now, we consider the solutions of (56). By Step 3, we get that,

$$L_i^2 + 3y^2 + 6z^2 - 6L_i yz - 1 = 0,$$

has solutions with $i \in \{1, 3\}$. If $i = 1$, we have the elliptic curves,

$$Y_1^2 = -180X_1^4 - 720X_1^2, \tag{63}$$
$$Y_1^2 = -180X_1^4 + 720X_1^2. \tag{64}$$

Equation (63) has no integer solutions, and $X_1 \in \{\pm 2\}$ is the X-coordinate of the solutions of (64). But, we notice that $\pm 2 \notin \{L_n\}$ for all $n \geq 1$, therefore, (56) has no solution at $i = 1$.

For $i = 3$, we obtain the elliptic curves,

$$Y_1^2 = 2520X_1^4 + 9180X_1^2 - 3600, \tag{65}$$
$$Y_1^2 = 2520X_1^4 - 10980X_1^2 + 3600. \tag{66}$$

Equation (65) has no integer solutions and $X_1 \in \{0, \pm 2, \pm 3\}$ is the X-coordinate of the solutions of (66). But, we notice that $0, \pm 2, \pm 3 \notin \{L_n\}$ for all $n \geq 3$. Therefore, (56) has no solution when $i = 3$.

Now, we study the solutions of (57). We follow Step 3 to eliminate the values of i. We obtain the equation,

$$L_i^2 + 6y^2 + 3z^2 - 6L_i yz - 1 = 0,$$

is solvable only with $i \in \{1, 3\}$. Next, we follow Step 4 to get values of j and k. For $i = 1$, we get the elliptic curves,

$$Y_1^2 = -180X_1^4 - 144X_1^2, \tag{67}$$
$$Y_1^2 = -180X_1^4 + 144X_1^2. \tag{68}$$

Equations (67) and (68) have no integer solutions. Therefore, (57) has no solution when $i = 1$.

Now, for $i = 3$, we get the elliptic curves,

$$Y_1^2 = 2520X_1^4 + 216X_1^2 - 1440, \tag{69}$$
$$Y_1^2 = 2520X_1^4 - 3816X_1^2 + 1440. \tag{70}$$

Using the same way to find $X_1 = L_k$, we have that only $X_1 \in \{\pm 1\}$. For $L_k = 1$, so $k = 1$. But, we noticed that $k = 1$. This is a contradiction as $3 = i \leq j \leq k$. Therefore, (57) has no solution when $i = 3$.

Also, we study the solutions of (58). We follow Step 3 to eliminate the values of i, we obtain the equation,

$$6L_i^2 + y^2 + 3z^2 - 6L_iyz - 1 = 0,$$

is solvable only with $i \in \{1, 2, 5, 7, 8\}$. Next, we follow Step 4 to get values of j and k. For $i = 1$, we get the elliptic curves,

$$Y_1^2 = 120X_1^4 + 380X_1^2 - 400, \tag{71}$$
$$Y_1^2 = 120X_1^4 - 580X_1^2 + 400. \tag{72}$$

With the Magma function `SIntegralLjunggrenPoints()`, we solve the above equations. We are indeed interested in the values of X_1 as $X_1 = L_k$. For (34), we get that $X_1 \in \{\pm 1, \pm 5, \pm 29\}$ for (71) and $X_1 \in \{0, \pm 2, \pm 3, \pm 7\}$ for (72). We notice that only $1, 3, 7, 29 \in \{L_n\}$ for all $n \geq 1$. Therefore, $L_k \in \{1, 3, 7, 29\}$. If $L_k = 1$, then $k = 1$. Now, we want to find the values of j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 1$. So, $(L_i, L_j, L_k) = (L_1, L_1, L_1) = (1, 1, 1)$ does not satisfy (58).
- In case of $j+1 \leq k \leq j+22$, then $j < 1$. This is not possible since $1 = i \leq j \leq k$.

If $L_k = 3$, then $k = 2$. From $j \leq k \leq j + 22$, we get the values of j as follows:

- If $k = j$, then $j = 2$. Therefore, $(L_i, L_j, L_k) = (L_1, L_2, L_2) = (1, 3, 3)$ and this triple does not satisfy (58).
- If $k = j + 1$, then $j = 1$. Therefore, $(L_i, L_j, L_k) = (L_1, L_1, L_2) = (1, 1, 3)$ and this $(1, 1, 3)$ does not satisfy (58).
- If $j + 2 \leq k \leq j + 22$, then $j < 1$. It is not possible since $1 = i \leq j \leq k$.

If $L_k = 7$, then $k = 4$. So, the values of j are obtained as follows:

- If $k = j$, then $j = 4$. So, $(L_i, L_j, L_k) = (L_1, L_4, L_4) = (1, 7, 7)$ which does not satisfy (58).
- If $k = j + 1$, then $j = 3$. Therefore, $(L_i, L_j, L_k) = (L_1, L_3, L_4) = (1, 4, 7)$ and (1,4,7) satisfy (58).
- If $k = j + 2$, then $j = 2$. We obtain that $(L_i, L_j, L_k) = (L_1, L_2, L_4) = (1, 3, 7)$ and this triple does not satisfy (58).
- If $k = j + 3$, then $j = 1$. However, $(L_i, L_j, L_k) = (L_1, L_1, L_4) = (1, 1, 7)$ does not satisfy (58).
- If $j + 4 \leq k = 4 \leq j + 22$, then $j < 1$. This is not possible since $1 = i \leq j \leq k$.

If $L_k = 29$, then $k = 7$. Now, we study the values of j :

- If $k = j$, then $j = 7$. So, $(L_i, L_j, L_k) = (L_1, L_7, L_7) = (1, 29, 29)$ which does not satisfy (58).

- If $k = j + 1$, then $j = 6$. But, $(L_i, L_j, L_k) = (L_1, L_6, L_7) = (1, 18, 29)$ does not satisfy (58).
- If $k = j + 2$, then $j = 5$. Also, $(L_i, L_j, L_k) = (L_1, L_5, L_7) = (1, 11, 29)$. This triple does not satisfy (58).
- If $k = j + 3$, then $j = 4$. Therefore, $(L_i, L_j, L_k) = (L_1, L_4, L_7) = (1, 7, 29)$ and this triple does not satisfy (58).
- If $k = j + 4$, then $j = 3$. However, $(L_i, L_j, L_k) = (L_1, L_3, L_7) = (1, 4, 29)$ does not satisfy (58).
- If $k = j + 5$, then $j = 2$. Thus, $(L_i, L_j, L_k) = (L_1, L_2, L_7) = (1, 3, 29)$ and this triple does not satisfy (58).
- If $k = j + 6$, then $j = 1$. Therefore, $(L_i, L_j, L_k) = (L_1, L_1, L_7) = (1, 1, 29)$ and this triple does not satisfy (58).
- If $j + 7 \leq k = 7 \leq j + 22$, then $j < 1$. It is not possible since $1 = i \leq j \leq k$.

Now, if $i = 2$, we get the elliptic curves,

$$Y_1^2 = 1560X_1^4 + 5180X_1^2 - 4240, \tag{73}$$
$$Y_1^2 = 1560X_1^4 - 7300X_1^2 + 4240. \tag{74}$$

For the curve (73), the possible values of X_1, that represent Lucas numbers are given by $X_1 = 1$. Also, with respect to the curve (74), we get that $X_1 = 2$. Therefore, we notice that only $L_n = 1$ for all $n \geq 1$. So, $L_k = 1$, then $k = 1$. But we noticed that $k < 2$. This is a contradiction with $2 = i \leq j \leq k$. Therefore, (58) has no solution when $i = 2$.

If $i \in \{5, 7, 8\}$, using the same way to find $X_1 = L_k$, we get that $X_1 \in \{\pm 1, \pm 2\}$ at $i = 5$, $X_1 = \pm 2$ at $i = 7$ and $X_1 \in \{\pm 2, \pm 3\}$ at $i = 8$. Therefore, we notice that these values of L_k leads to $k \leq 2$. But, this is a contradiction since $5 \leq i \leq j \leq k$.

Also, we study the solutions of (59). We follow Step 3 to eliminate the values of i. We obtain the equation,

$$6L_i^2 + 3y^2 + z^2 - 6L_i yz - 1 = 0,$$

is solvable only with $i \in \{1, 2, 5, 7, 8\}$. Next, we follow Step 4 to get values of j and k. For $i = 1$, we get the elliptic curves,

$$Y_1^2 = 120X_1^4 + 180X_1^2 - 1200, \tag{75}$$
$$Y_1^2 = 120X_1^4 - 780X_1^2 + 1200. \tag{76}$$

By the Magma function `SIntegralLjunggrenPoints()`, we get that $X_1 = \pm 4$ for (75) and $X_1 = \pm 2$ for (76). We notice that $4 \in \{L_n\}$ for all $n \geq 1$. If $L_k = 4$, then $k = 3$. Now, we want to find the values of j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 3$. So, $(L_i, L_j, L_k) = (L_1, L_3, L_3) = (1, 4, 4)$ and this triple does not satisfy (59).
- If $k = j + 1$, then $j = 2$. Therefore, $(L_i, L_j, L_k) = (L_1, L_2, L_3) = (1, 3, 4)$ and $(1, 3, 4)$ does not satisfy (59).
- If $k = j + 2$, then $j = 1$. So, $(L_i, L_j, L_k) = (L_1, L_1, L_3) = (1, 1, 4)$ and this triple satisfy (59).
- If $j + 3 \leq k \leq j + 22$ then $j < 1$. It is not possible since $1 = i \leq j \leq k$.

If $i = 2$, we get the elliptic curves,

$$Y_1^2 = 1560X_1^4 + 3060X_1^2 - 12720, \tag{77}$$
$$Y_1^2 = 1560X_1^4 - 9420X_1^2 + 12720. \tag{78}$$

For (77), we get that $X_1 = \pm 4$, and $X_1 = \pm 2$ for (78). We notice that $4 \in \{L_n\}$ for all $n \geq 1$. If $L_k = 4$, then $k = 3$. Now, we want to find of values the j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 3$. So, $(L_i, L_j, L_k) = (L_2, L_3, L_3) = (3, 4, 4)$ and this triple does not satisfy (59).
- If $k = j + 1$, then $j = 2$. Therefore, $(L_i, L_j, L_k) = (L_2, L_2, L_3) = (3, 3, 4)$ and $(3, 3, 4)$ does not satisfy (59).
- If $k \in \{j + 2, \ldots, j + 22\}$, then $j \leq 1$. This is not possible since $2 = i \leq j \leq k$.

Finally, if $i \in \{5, 7, 8\}$, using the same way to find $X_1 = L_k$, we get that $X_1 = \pm 2$ at $i = 5$, $X_1 = \pm 2$ at $i = 7$ and $X_1 = \pm 2$ at $i = 8$. But, $\pm 2 \notin \{L_n\}$ for all $n \geq 1$. Therefore, (59) has no solution when $i \in \{5, 7, 8\}$.

Also, we study the solutions of (60). We follow Step 3 to eliminate the values of i. We obtain the equation,

$$3L_i^2 + 6y^2 + z^2 - 6L_i yz - 1 = 0,$$

is solvable only with $i \in \{1, 2, 4, 7\}$. Next, we follow Step 4 to get values of j and k. For $i = 1$, we get the elliptic curves,

$$Y_1^2 = 60X_1^4 - 960, \tag{79}$$
$$Y_1^2 = 60X_1^4 - 480X_1^2 + 960. \tag{80}$$

By the Magma function `SIntegralLjunggrenPoints()`, we get that $X_1 \in \{\pm 2, \pm 4\}$ for (79) and $X_1 = \pm 2$ for (80), we notice that only $4 \in \{L_n\}$ as $n \geq 1$. If $L_k = 4$, then $k = 3$. Now, we want to find the values of j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 3$. So, $(L_i, L_j, L_k) = (L_1, L_3, L_3) = (1, 4, 4)$ and this triple does not satisfy (60).
- If $k = j + 1$, then $j = 2$. Therefore, $(L_i, L_j, L_k) = (L_1, L_2, L_3) = (1, 3, 4)$ and this $(1, 3, 4)$ satisfy (60).
- If $k = j + 2$, then $j = 1$. So, we have $(L_i, L_j, L_k) = (L_1, L_1, L_3) = (1, 1, 4)$ and this triple does not satisfy (60).
- If $j + 3 \leq k = 3 \leq j + 22$, then $j < 1$. It is not possible since $1 = i \leq j$.

Similarly, for $i = 2$, we obtain the elliptic curves,

$$Y_1^2 = 1500X_1^4 + 2880X_1^2 - 12480, \tag{81}$$
$$Y_1^2 = 1500X_1^4 - 9120X_1^2 + 12480. \tag{82}$$

Equation (81) has no integer solutions and $X_1 \in \{\pm 2\}$ is the X-coordinate of the solutions of (82). But, we notice that $\pm 2 \notin \{L_n\}$ for all $n \geq 1$. Therefore, (60) has no solution.

Finally, if $i \in \{4, 7\}$, using the same way to find $X_1 = L_k$, we get that only $X_1 \in \{\pm 2, \pm 4\}$ for $i = 4$ and $X_1 = \pm 2$ for $i = 7$. We get that only $4 \in \{L_n\}$ for all $n \geq 1$. So, $L_k = 4$, then $k = 3$. This is a contradiction with $4 = i \leq j \leq k$. Therefore, (60) has no solution when $i = 4, 7$.

By gathering all of the obtained solutions of (55)–(60) with permuting their components so that they satisfy (10), we get the solutions of (10) as follows:

$$(L_i, L_j, L_k) \in \{(1,4,1), (1,4,3), (7,4,1)\}.$$

Case 4: If $(A, B, C, D) = (5, 1, 5, 5)$. The distinct equations derived (11) are:

$$5L_i^2 + L_j^2 + 5L_k^2 = 5L_iL_jL_k + 1, \tag{83}$$
$$L_i^2 + 5L_j^2 + 5L_k^2 = 5L_iL_jL_k + 1, \tag{84}$$
$$5L_i^2 + 5L_j^2 + L_k^2 = 5L_iL_jL_k + 1, \tag{85}$$

where $1 \leq i \leq j \leq k$. From Theorem 3.1, we get that $i \leq 13$ and $j \leq k \leq j + 22$ in all (83)–(85). Now, we study the solutions of (83). By Step 3, we get that,

$$5L_i^2 + y^2 + 5z^2 - 5L_iyz - 1 = 0,$$

is solvable only with $i \in \{1, 3, 4, 5\}$. By Step 4, if $i = 1$, we substitute $L_i = 1$ and $(a, b, c, d) = (5, 1, 5, 5)$ in (29), we obtain that,

$$Y_1^2 = 25X_1^4 + 20X_1^2 - 320, \tag{86}$$
$$Y_1^2 = 25X_1^4 - 180X_1^2 + 320. \tag{87}$$

We obtain that $X_1 = \pm 4$ for (86) and $X_1 = \pm 2$ for (87). We notice that only $L_n = 4$ for all $n \geq 1$. If $L_k = 4$, then $k = 3$. Now, we want to find all the values of j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 3$. Therefore, $(L_i, L_j, L_k) = (L_1, L_3, L_3) = (1, 4, 4)$ and this triple does not satisfy (83).
- If $k = j + 1$, then $j = 2$. So, $(L_i, L_j, L_k) = (L_1, L_2, L_3) = (1, 3, 4)$, and $(1, 3, 4)$ does not satisfy (83).
- If $k = j + 2$, then $j = 1$. However, $(L_i, L_j, L_k) = (L_1, L_1, L_3) = (1, 1, 4)$ does not satisfy (83).
- If $j + 3 \leq k \leq j + 22$, then $j < 1$. It is not possible since $j \geq 1$.

If $i \in \{3, 4, 5\}$, using the same way to find $X_1 = L_k$, we get that $X_1 \in \{\pm 1, \pm 2\}$ for $i = 3$, $X_1 = \pm 2$ for $i = 4$ and $X_1 = \pm 2$ for $i = 5$. We notice that the values of L_k leads to $k \leq 1$. Therefore, this is a contradiction since $3 \leq i \leq j \leq k$. Therefore, (83) has no solution when $i \in \{3, 4, 5\}$.

Next, we study the solutions of (84). We follow Step 3 to eliminate the values of i. We obtain the equation,

$$L_i^2 + 5y^2 + 5z^2 - 5L_iyz - 1 = 0,$$

is solvable with $i \in \{1, 3\}$. Next, we follow Step 4 to get values of j and k. For $i = 1$, we get the elliptic curves,

$$Y_1^2 = -375X_1^4 - 1500X_1^2, \tag{88}$$
$$Y_1^2 = -375X_1^4 + 1500X_1^2. \tag{89}$$

Using the same way to find $X_1 = L_k$, we get that (88) has no integer solutions, and $X_1 = \pm 2$ for (89). But, we notice that $\pm 2 \notin \{L_n\}$ for all $n \geq 1$. Therefore, (84) has no solution when $i = 1$.

Now if $i = 3$, we obtain the elliptic curves,

$$Y_1^2 = 1500X_1^4 + 4500X_1^2 - 6000, \tag{90}$$
$$Y_1^2 = 1500X_1^4 - 7500X_1^2 + 6000. \tag{91}$$

We obtain that $X_1 = \pm 1$ for (90) and $X_1 \in \{\pm 1, \pm 2, \pm 7\}$ for (91). We notice that only $L_k = 7$ as $k \geq 3$. If $L_k = 7$, then $k = 4$. Now, we want to find the values of j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 4$. So,$(L_i, L_j, L_k), (L_3, L_4, L_4) = (4, 7, 7)$ and this $(4, 7, 7)$ does not satisfy (84).
- If $k = j + 1$, then $j = 3$. Therefore, $(L_i, L_j, L_k) = (L_3, L_3, L_4) = (4, 4, 7)$ and this triple does not satisfy (84).
- If $k \in \{j + 2, \ldots, j + 22\}$, then $j \leq 2$, which is not possible since $3 = i \leq j \leq k$.

Finally, we study the solutions of (85). By Step 3 we get that equation,

$$5L_i^2 + 5y^2 + z^2 - 5L_i yz - 1 = 0,$$

is solvable if $i \in \{1, 3, 4, 5\}$. Now, we follow Step 4. If $i = 1$, we substitute $L_i = L_1 = 1$ and $(a, b, c, d) = (5, 5, 1, 5)$ in (29), we obtain that,

$$Y_1^2 = 25X_1^4 - 300X_1^2 - 1600, \tag{92}$$
$$Y_1^2 = 25X_1^4 - 500X_1^2 + 1600. \tag{93}$$

By the Magma function `SIntegralLjunggrenPoints()`, we get that $X_1 = \pm 4$ for (92) and $X_1 \in \{\pm 2, \pm 4\}$ for (93). We notice that only $L_n = 4$ for all $n \geq 1$. Therefore, we get that only $L_k = 4$, that implies $k = 3$. Now, we want to find the values of j from $j \leq k \leq j + 22$ as follows:

- If $k = j$, then $j = 3$. We get that $(L_i, L_j, L_k) = (L_1, L_3, L_3) = (1, 4, 4)$ and this triple does not satisfy (85).
- If $k = j + 1$, then $j = 2$. Therefore, $(L_i, L_j, L_k) = (L_1, L_2, L_3) = (1, 3, 4)$ which does not satisfy (85).
- If $k = j + 2$, then $j = 1$. So, $(L_i, L_j, L_k) = (L_1, L_1, L_3) = (1, 1, 4)$ and this $(1, 1, 4)$ does not satisfy (85).
- If $j + 3 \leq k \leq j + 22$, then $j < 1$. It is not possible since $j \geq 1$.

Similarly, for $i = 3$, we obtain the elliptic curves,

$$Y_1^2 = 1900X_1^4 - 300X_1^2 - 31600, \tag{94}$$
$$Y_1^2 = 1900X_1^4 - 15500X_1^2 + 31600. \tag{95}$$

Equation (94) has no integer solutions and $X_1 \in \{\pm 2\}$ is the only X-coordinate of the solutions of (95). But, we notice that $\pm 2 \notin \{L_n\}$ for all $n \geq 1$. Therefore, (85) has no solution at $i = 3$.

If $i \in \{4, 5\}$, using the same way to find $X_1 = L_k$, we get that $X_1 \in \{\pm 2, \pm 4\}$ for $i = 4$ and $X_1 = \pm 2$ with $i = 5$. We notice that the values of L_k leads to $k \leq 3$ and this is a contradiction since $4 \leq i \leq j \leq k$. Hence, we conclude that (85) has no solution when $i \in \{4, 5\}$.

As a result, we get that (83)–(85) do not have solutions. Therefore, (11) has no solution in Lucas numbers.

Case 5: If $(A, B, C, D) = (6, 10, 15, 30)$. The distinct equations derived (12) are:

$$6L_i^2 + 10L_j^2 + 15L_k^2 = 30L_iL_jL_k + 1, \tag{96}$$
$$10L_i^2 + 6L_j^2 + 15L_k^2 = 30L_iL_jL_k + 1, \tag{97}$$
$$6L_i^2 + 15L_j^2 + 10L_k^2 = 30L_iL_jL_k + 1, \tag{98}$$
$$15L_i^2 + 6L_j^2 + 10L_k^2 = 30L_iL_jL_k + 1, \tag{99}$$
$$10L_i^2 + 15L_j^2 + 6L_k^2 = 30L_iL_jL_k + 1, \tag{100}$$
$$15L_i^2 + 10L_j^2 + 6L_k^2 = 30L_iL_jL_k + 1, \tag{101}$$

where $1 \le i \le j \le k$. From Theorem 3.1, we get that $i \le 13$ and $j \le k \le j + 22$ in all (96)–(101). Now, we study the solutions of (96) in detail and the remainder equations are studied in the same way. So, the details of computations for determining the complete set of solutions for these equations are omitted. We follow Step 3 to eliminate the values of i. We get that,

$$6L_i^2 + 10y^2 + 15z^2 - 30L_iyz - 1 = 0,$$

is solvable only with $i \in \{1, 3, 7\}$. Next, we follow Step 4 to get the values of j and k. For $i = 1$, we obtain the elliptic curves,

$$Y_1^2 = 1500X_1^4 + 5000X_1^2 - 4000, \tag{102}$$
$$Y_1^2 = 1500X_1^4 - 7000X_1^2 + 4000. \tag{103}$$

We have that $X_1 \in \{\pm 1, \pm 2, \pm 11\}$ for (102) and $X_1 \in \{\pm 2, \pm 3\}$ for (103). We notice that only $1, 3, 11 \in \{L_n\}$ for all $n \ge 1$. If $L_k = 1$, then $k = 1$. Now, we want to find the values of j from $j \le k \le j + 22$ as follows:

- If $k = j$, then $j = 1$. Hence, $(L_i, L_j, L_k) = (L_1, L_1, L_1) = (1, 1, 1)$ and this triple satisfy (96).
- In case of $j + 1 \le k = 1 \le j + 22$, so $j < 1$. That is not possible since $1 = i \le j \le k$.

If $L_k = 3$, then $k = 2$. From $j \le k \le j + 22$, we get that the values of j as follows:

- If $k = j$, then $j = 2$. But, $(L_i, L_j, L_k) = (L_1, L_2, L_2) = (1, 3, 3)$ does not satisfy (96).
- If $k = j + 1$, then $j = 1$. Therefore, $(L_i, L_j, L_k) = (L_1, L_1, L_2) = (1, 1, 3)$ and (96) does not hold at this triple.
- In case of $j + 2 \le k \le j + 22$, then $j < 1$. This is not possible since $j \ge 1$.

If $L_k = 11$, then $k = 5$. In a similar way to done earlier, after determining the values of j and k, we get no solutions to (96) other than the case where $5 = k = j + 1$. Indeed, here we get the solution $(L_i, L_j, L_k) = (L_1, L_4, L_5) = (1, 7, 11)$. It remains to investigate the values of j and k correspondingly the values of i with $i \in \{3, 7\}$. By applying Step 4, we get the elliptic curves,

$$Y_1^2 = 69000_1^4 + 257000X_1^2 - 76000, \tag{104}$$
$$Y_1^2 = 69000X_1^4 - 295000X_1^2 + 76000, \tag{105}$$

and

$$Y_1^2 = 3781500X_1^4 + 14117000X_1^2 - 4036000, \tag{106}$$
$$Y_1^2 = 3781500X_1^4 - 16135000X_1^2 + 4036000, \tag{107}$$

for $i = 3$ and $i = 7$, respectively. From the Magma function `SIntegralLjunggrenPoints()`, we obtain no solutions to these curves. Following the same techniques on (97)–(101), we obtain:

$$(L_i, L_j, L_k) \in \{(1,1,1),(1,4,7)\}, \quad (L_i, L_j, L_k) \in \{(1,1,1),(1,3,7)\},$$
$$(L_i, L_j, L_k) \in \{(1,1,1),(1,4,11)\}, \quad (L_i, L_j, L_k) \in \{(1,1,1),(1,1,4)\},$$
$$\text{and} \quad (L_i, L_j, L_k) \in \{(1,1,1),(1,1,4)\},$$

as solutions to (97), (98), (99), (100) and (101), respectively. From the above solution, we notice that (96)–(101) contain solutions, and by permuting these solutions, we obtain solutions to (12) as follows:

$$(L_i, L_j, L_k) \in \{(1,1,1),(1,7,3),(1,7,11),(4,1,1),(4,1,7),(4,11,1)\}.$$

Case 6: If $(A, B, C, D) = (7, 2, 14, 14)$. The distinct equations derived (13) are:

$$7L_i^2 + 2L_j^2 + 14L_k^2 = 14L_iL_jL_k + 1, \tag{108}$$
$$2L_i^2 + 7L_j^2 + 14L_k^2 = 14L_iL_jL_k + 1, \tag{109}$$
$$7L_i^2 + 14L_j^2 + 2L_k^2 = 14L_iL_jL_k + 1, \tag{110}$$
$$14L_i^2 + 7L_j^2 + 2L_k^2 = 14L_iL_jL_k + 1, \tag{111}$$
$$14L_i^2 + 2L_j^2 + 7L_k^2 = 14L_iL_jL_k + 1, \tag{112}$$
$$2L_i^2 + 14L_j^2 + 7L_k^2 = 14L_iL_jL_k + 1, \tag{113}$$

where, $1 \le i \le j \le k$. From Theorem 3.1, we get that $i \le 13$ and $k \le j+22$ in all (108)–(113). Furthermore, we get that $i \in \{1, 2\}$ in (108) and (110). Also $i \in \{1, 3, 7, 13\}$ in (111) and (112). But, for (109) and (113) have no integer solutions y and z in which (109) and (113) are satisfied. Using Step 4 leads to (108)–(113) have no solutions of the form (L_i, L_j, L_k). So, (13) has no solution of the form (L_i, L_j, L_k) with $i, j, k \ge 1$.

Therefore, Theorem 3.2 is completely proved.

□

4 Conclusion

As the Jin-Schmidt equation has infinitely many solutions in positive integers, the result of this paper implies that this equation has a finite number of solutions in the sequence of Lucas numbers. This result will give a deep insight in the study of Diophantine approximations, as the solutions of Markoff equation and its generalizations are connected to Diophantine approximations.

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